Announcements

1) HW #4 due Sunday

2) Quiz on Tuesday

Covering sections 15.3 and 15.4, practice problems on Canvas

Integration Over More General Regions

(Section 15.3)

Assume a region R is bounded in $\mathbb{R}^2$

(there is a number $r > 0$

such that R is contained

in a circle of radius r

centered at the origin)

The circle is contained in

the square $[-r, r] \times [-r, r]$

The Definition

Let R be a bounded region in $\mathbb{R}^2$ and let $z = f(x,y)$ be continuous on R. R is contained in the square $[-r,r] \times [-r,r]$ for some $r > 0$.

Define $\boxed{g(x,y) = \begin{cases} f(x,y), & (x,y) \text{ in } R \\ 0, & (x,y) \text{ not in } R \end{cases}}$

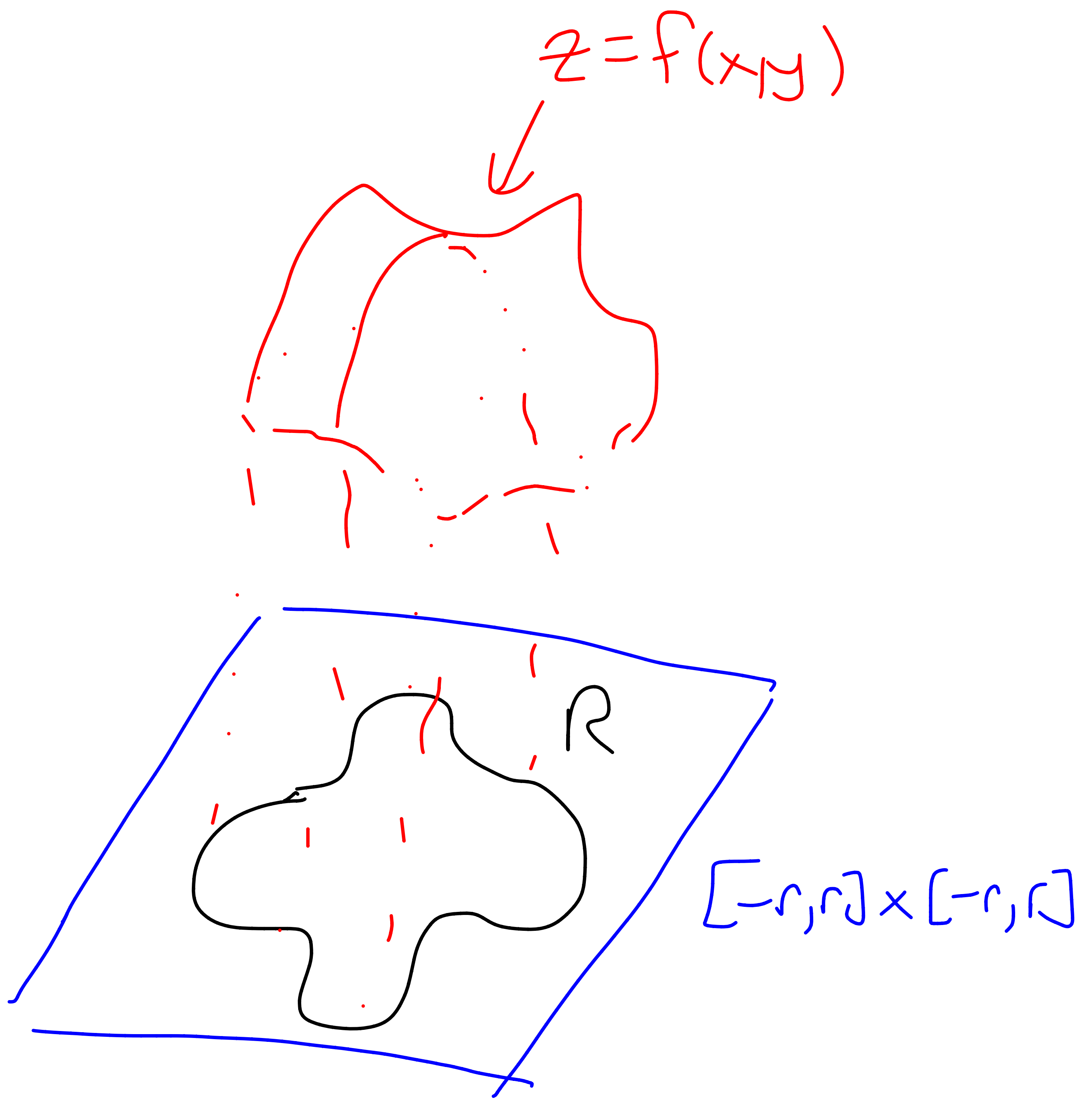
We define

$$\boxed{\iint_R f(x,y)\,dA = \iint_{[-r,r]\times[-r,r]} g(x,y)\,dA}$$

provided the integral of g exists.

Note: g is probably discontinuous, but its discontinuities are usually over a one- or zero-dimensional region, so the integral will be zero there.

Picture



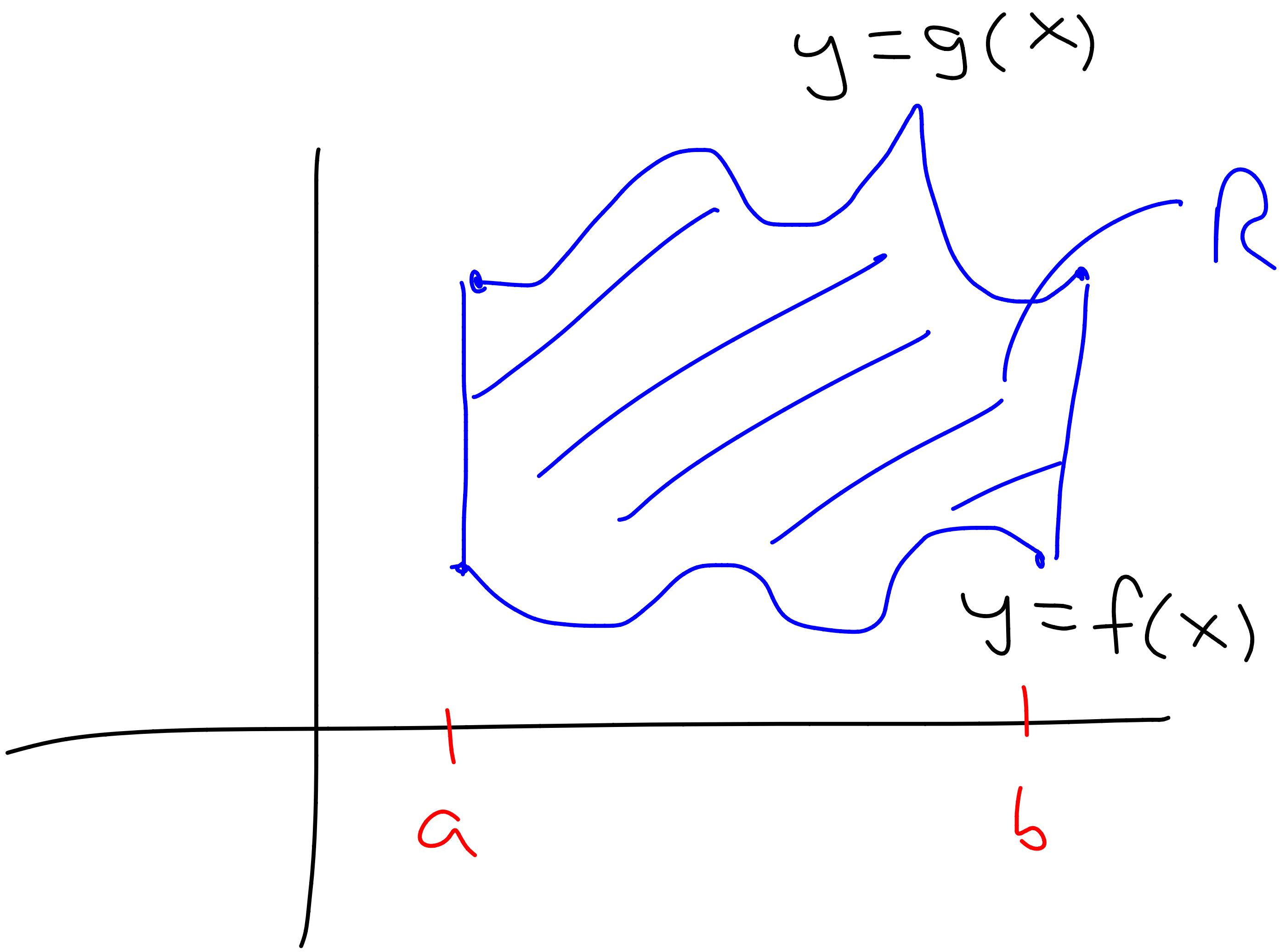
g equals f on R, zero on the rest of the square

Type I Regions

A bounded region R in $\mathbb{R}^2$ is called Type I if there are numbers a and b and functions f and g of one variable such that (x,y) is in R if

$$\boxed{a \le x \le b \,, \; f(x) \le y \le g(x)}$$

Picture



Notation: When we write

$$\{ A \mid B \}$$, it means

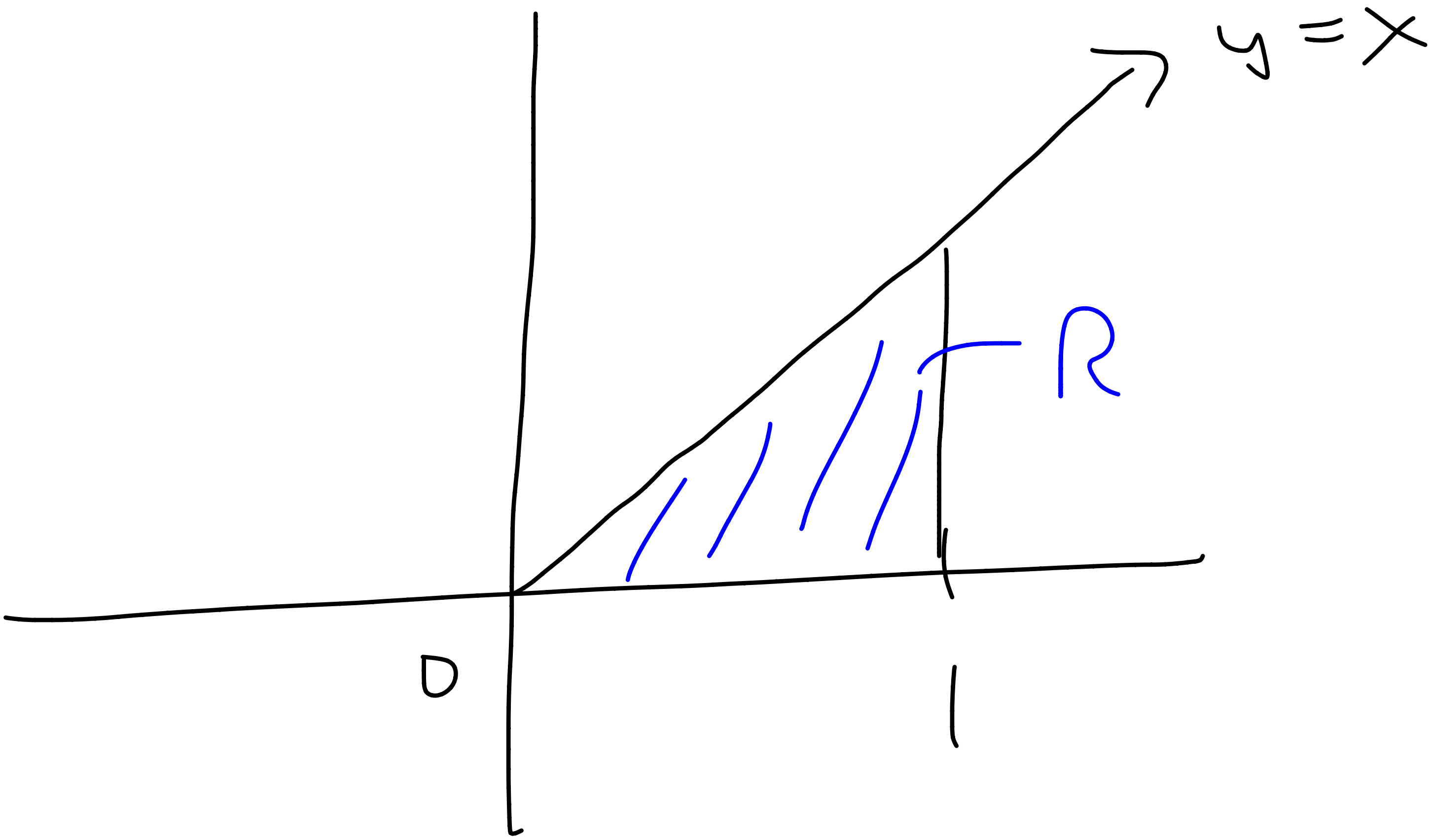
"the set of all A such that B"

Example 1:

$$\iint_R (x-y)\,dA$$

where $R=\{(x,y) \mid 0 \le x \le 1,\ \underbrace{0}_{f(x)} \le y \le \underbrace{x}_{g(x)}\}$

Draw R



Using Fubini,

$$\iint_R (x-y)\,dA = \underbrace{\int_0^1}_{\text{x-bounds}} \left(\underbrace{\int_0^x}_{\text{y-bounds}} (x-y)\,dy \right) dx$$

$$= \int_0^1 \left(xy - \frac{y^2}{2} \right) \Big|_0^x \, dx$$

$$= \int_0^1 \left(x^2 - \frac{x^2}{2} \right) dx$$

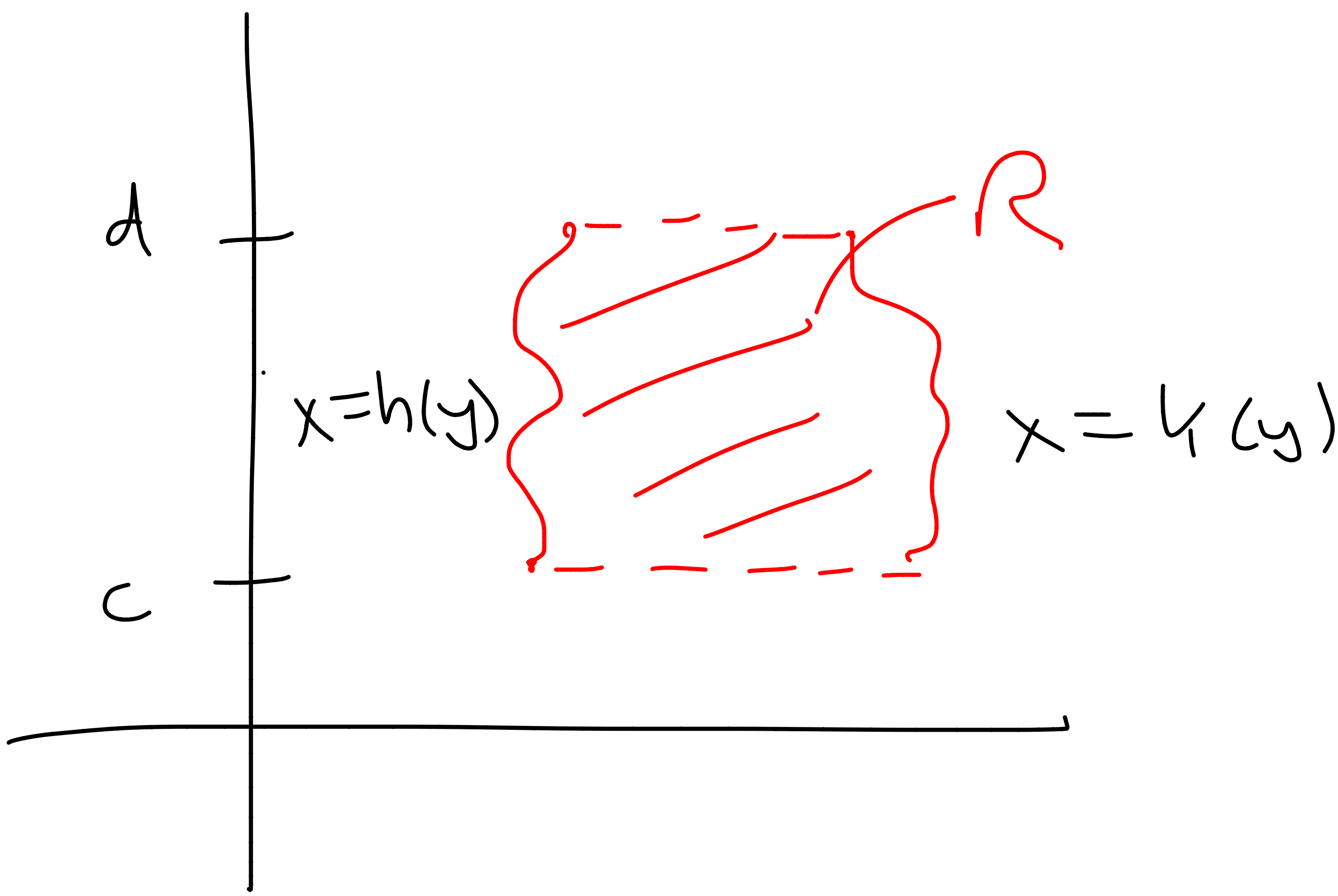
$$= \int_0^1 \frac{x^2}{2}\,dx = \frac{x^3}{6} \Big|_0^1 = \boxed{\frac{1}{6}}$$

Type II Regions

A bounded region R in $\mathbb{R}^2$ is said to be of Type II if there are numbers c and d and functions h and k with

$$\boxed{R = \{(x,y) \mid c \leq y \leq d,\ h(y) \leq x \leq k(y)\}}$$

Picture

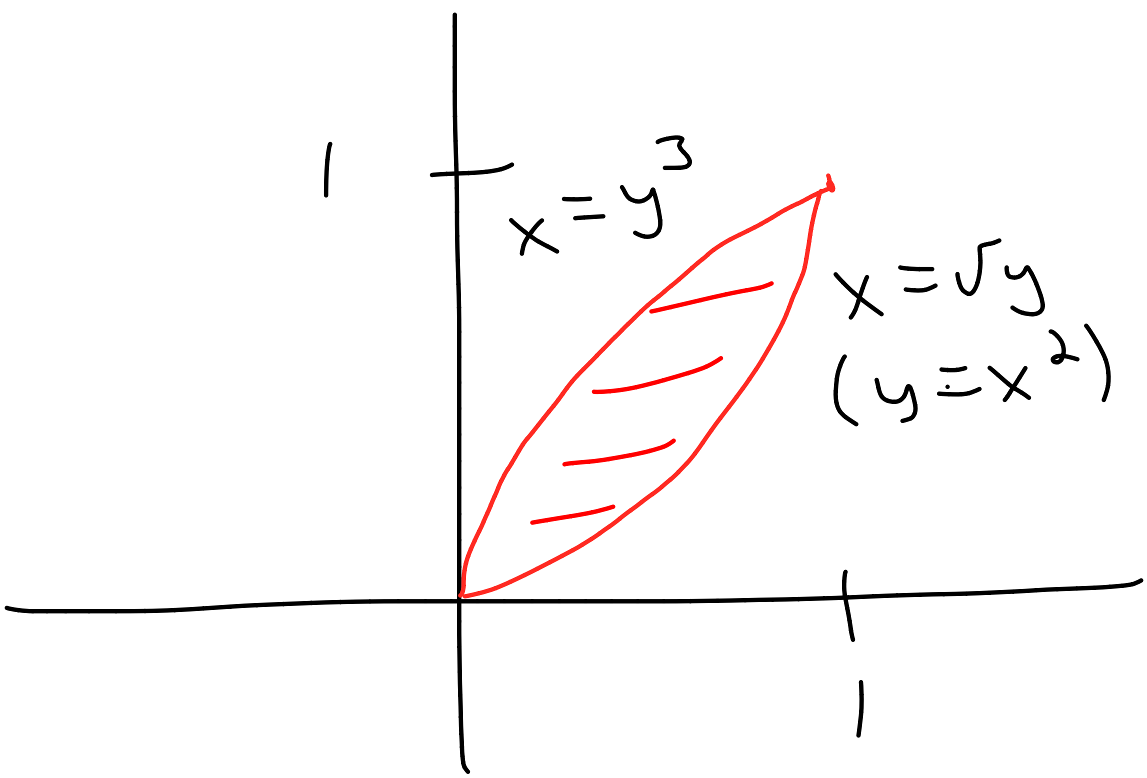


Example 2: $\iint_R (x^2+y^2)\,dA$

where

$$R = \{(x,y) \mid 0 \le y \le 1,\ y^3 \le x \le \sqrt{y}\}$$

Draw R



By Fubini,

$$\iint_R (x^2+y^2)\,dA$$

$$= \int_0^1 \left(\int_{y^3}^{\sqrt{y}} (x^2+y^2)\,dx \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + y^2 x \Big|_{y^3}^{\sqrt{y}} \right) dy$$

$$= \int_0^1 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^9}{3} - y^5 \right) dy$$

$$\int_0^1 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^9}{3} - y^5 \right) dy$$

$$= \left(\frac{2y^{5/2}}{15} + \frac{2y^{7/2}}{7} - \frac{y^{10}}{30} - \frac{y^6}{6} \right) \Bigg|_0^1$$

$$= \boxed{\frac{2}{15} + \frac{2}{7} - \frac{1}{30} - \frac{1}{6}}$$

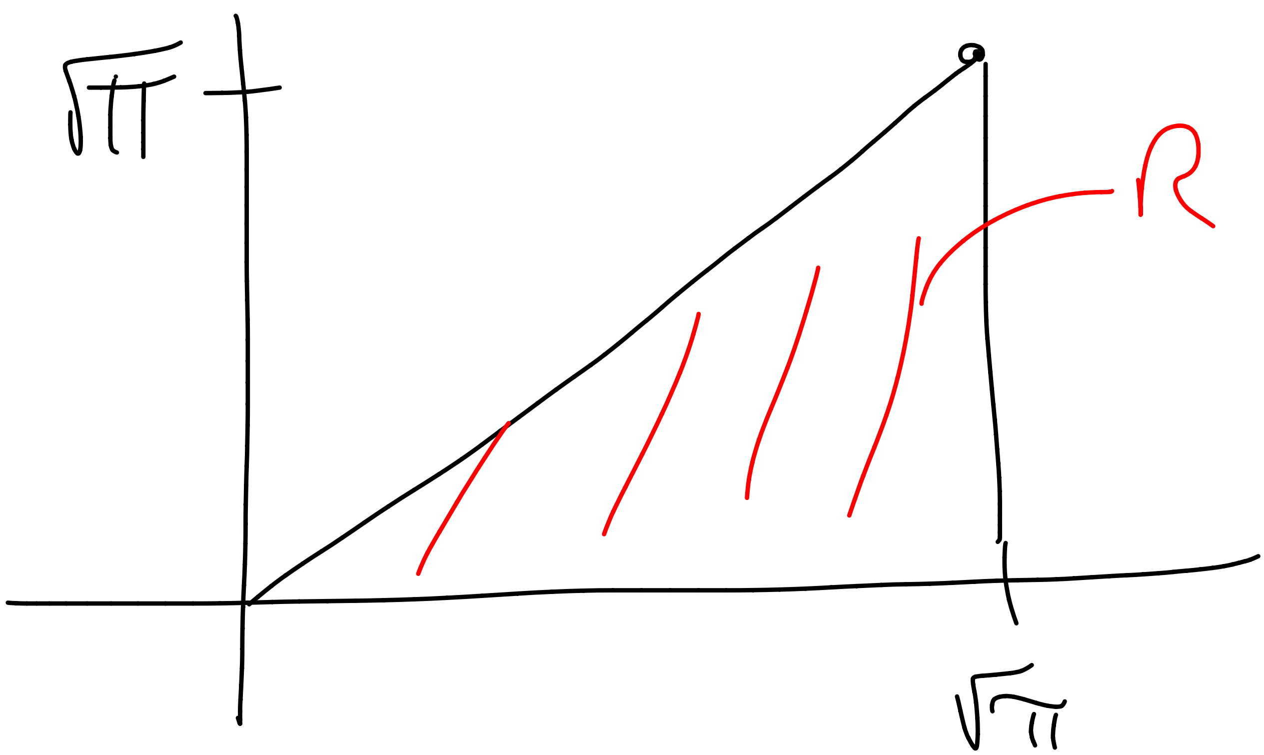
The most frustrating regions are both type I and type II – you need to choose an order, and sometimes, the order matters!

Example 3: $\iint_R \cos(x^2)\,dA$

where

$$R = \{(x,y) \mid 0 \le y \le x,\ 0 \le x \le \sqrt{\pi}\}$$

Draw R



$R = \{(x,y) \mid 0 \le y \le x,\ 0 \le x \le \sqrt{\pi}\}$

$= \{(x,y) \mid 0 \le y \le \sqrt{\pi},\ y \le x \le \sqrt{\pi}\}$

Can choose which direction to integrate in! I don't know $\int \cos(x^2)\,dx$, try y first:

$$\iint_R \cos(x^2)\,dx = \int_0^{\sqrt{\pi}} \left(\int_0^x \cos(x^2)\,dy \right) dx$$

$$\int_0^{\sqrt{\pi}}\left(\int_0^x \cos(x^2)\,dy\right)dx$$

$$= \int_0^{\sqrt{\pi}} \left(y\cos(x^2)\right)\Big|_0^x \,dx$$

$$= \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$

$u = x^2$, $du = 2x\,dx$,

So the integral becomes

$$= \frac{1}{2}\int_0^{\pi} \cos(u)\,du = \frac{\sin(u)}{2}\Big|_0^{\pi}$$

$$= \boxed{0}$$